THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
SAYT1134 Towards Differential Geometry
Tutorial 9 (August 27)

1 Shape Operator (or Weingarten Map)

1.1 Definition

Let S be a regular surface in $\mathbb{R}^3$ with regular parametrization $\mathbf{x}(u, v)$. For each point $p \in S$, define a linear map $\mathcal{S}_p : T_pS \to T_pS$ called the ***shape operator*** of S by **the negative differential of Gauss map**, that is

$$\mathcal{S}_p \triangleq -d\mathbf{n}_p$$

1.2 Matrix Representation

As the shape operator is a linear operator on T_pS, so it has a matrix representation to the basis $\mathbf{x}_u, \mathbf{x}_v$ as well, which is given by

$$\begin{aligned}\mathcal{S}_p = (II)(I)^{-1} &= \frac{1}{EG - F^2}\begin{pmatrix} e & f \\ f & g \end{pmatrix}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \\ &= \frac{1}{EG - F^2}\begin{pmatrix} eG - fF & fE - eF \\ fG - gF & gE - fF \end{pmatrix}\end{aligned}$$

1.3 Self-adjointness of Shape Operator

The shape operator $\mathcal{S}_p : T_pS \to T_pS$ is self-adjoint, meaning symmetric in matrix. In other words, for any $\mathbf{u}, \mathbf{v} \in T_pS$, we have

$$\langle \mathcal{S}_p(\mathbf{u}), \mathbf{v} \rangle = \langle \mathbf{u}, \mathcal{S}_p(\mathbf{v}) \rangle$$

1.4 Curvature from the Shape Operator

1.4.1 Eigenvalue and Eigenvector

From the above matrix representation, let $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in T_pS$ be two linearly independent eigenvectors of the shape operator S_p at p and κ_1, κ_2 be the associated eigenvalues respectively, that is

$$\begin{cases} \mathcal{S}_p(\boldsymbol{\xi}_1) = \kappa_1 \boldsymbol{\xi}_1 \\ \mathcal{S}_p(\boldsymbol{\xi}_2) = \kappa_2 \boldsymbol{\xi}_2 \end{cases}$$

Then, we say that κ_1, κ_2 are the ***principal curvatures*** of S at p, and $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$ are the corresponding ***principal directions***.

1.4.2 Gaussian and Mean Curvature

Let $S = \mathbf{x}(u, v)$ be a regular parametrized surface with the shape operator $\mathcal{S}_p$ at the point p. Then, the **Gaussian curvature** and the **Mean curvature** of S at p are given by

$$\begin{aligned} K(p) &= \det(\mathcal{S}_p) = \det((II)(I)^{-1}) = \frac{eg - f^2}{EG - F^2} \\ H(p) &= \frac{1}{2}\mathrm{tr}(\mathcal{S}_p) = \frac{1}{2}\mathrm{tr}((II)(I)^{-1}) = \frac{1}{2}\left(\frac{gE - 2fF + eG}{EG - F^2}\right) \end{aligned}$$

respectively, where I and II denote the first and the second fundamental forms of $\mathbf{x}$ as 2×2 matrices respectively. These give us a characteristic polynomial of the shape operator, that is

$$\mathcal{S}_p^2 - 2H(p)\mathcal{S}_p + K(p)\mathrm{id} = \mathbf{0}$$

Please refer to the Question 5 of the Exercise 6 for more details.

1.5 Relation between Gaussian curvature and Mean curvature & Principal curvatures

Since $\mathcal{S}_p$ is self-adjoint and hence diagonalizable. From subsection 1.4.1, the eigenvalues of $\mathcal{S}_p$ are the principal curvatures κ_1 and κ_2. Also, from the first part of the content of tutorial 8, we have

$$\begin{cases} K(p) = \det(\mathcal{S}_p) = \kappa_1\kappa_2 \\ H(p) = \dfrac{1}{2}\mathrm{tr}(\mathcal{S}_p) = \dfrac{\kappa_1 + \kappa_2}{2} \end{cases}$$

2 Isometry

2.1 Definition

Let S_1 and S_2 be regular surfaces. Let $f : S_1 \to S_2$ be a differentiable bijective map from S_1 to S_2. We say that a map $f : S_1 \to S_2$ is an **isometry** if $I_1(u, v) = I_2(u, v)$ for any (u, v), where $I_1(u, v)$ and $I_2(u, v)$ are the first fundamental forms of S_1 and $S_2 = f(S_1)$. Also, we say S_1 and S_2 are **isometric** if there is an isometry between S_1 and S_2.

2.2 Theorema Egregium

If S_1 and S_2 are isometric, say there is an isometry $f : S_1 \to S_2$ between S_1 and S_2. Then for any $p \in S_1$, the Gaussian curvature of S_1 at p is equal to the Gaussian curvature of S_2 at $f(p)$, that is

$$K(p) = K\left(f(p)\right)$$

for any $p \in S_1$.
Warning: The converse of the above does not hold.

2.2.1 Example

1. Let $S_1 = \{z = 0\}$ be the xy-plane and $S_2 = \{x^2 + y^2 = 1\}$ be the right unit cylinder. Show that the map $f : S_1 \to S_2$ defined by $f(x, y, 0) = (\cos x, \sin x, y)$ is a local isometry.

2. (Area Preserving) Let $f : S_1 \to S_2$ be an isometry between two bounded and closed surfaces S_1 and S_2 in $\mathbb{R}^3$. Show that S_1 and S_2 have the same area.

Solution.

1. Parametrize S_1 by $\mathbf{u}_1(x, y) = (x, y, 0)$ and S_2 by $\mathbf{u}_2(x, y) = f(\mathbf{x}_1(u, v)) = (\cos x, \sin x, y)$. Then, we have

$$\begin{aligned} (\mathbf{u}_1)_x &= (1, 0, 0) \\ (\mathbf{u}_1)_y &= (0, 1, 0) \\ (\mathbf{u}_2)_x &= (-\sin x, \cos x, 0) \\ (\mathbf{u}_2)_y &= (0, 0, 1) \end{aligned}$$

Therefore, the first fundamental form of S_1 and S_2 are

$$I_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_1$$

for any $(x, y) \in \mathbb{R}^2$ Thus, f is a local isometry.

2. Let $\mathbf{x} : (u, v) \in D \to S_1$ be a parametrization of S_1.
   Then, $f \circ \mathbf{x} : D \to S_2$ is a parametrization of S_2 except at a set of measure zero.
   By the assumption that f is an isometry, so $I_1(u, v) = I_2(u, v)$ for any $(u, v) \in D$, where I_1, I_2 denote the first fundamental form of S_1 and S_2 respectively.
   Thus, we have

$$\begin{aligned} \text{Area}(S_1) &= \int_D \sqrt{\det(I_1(u, v))}\, dudv \\ &= \int_D \sqrt{\det(I_2(u, v))}\, dudv \\ &= \text{Area}(S_2) + 0 \\ &= \text{Area}(S_2) \end{aligned}$$

A typical example for surfaces is Catenoid and Helicoid under the following parametrization:

- **(Catenoid)** $\mathbf{x}_1(\theta, v) = (\cosh v \cos\theta, \cosh v \sin\theta, v),\ (\theta.v) \in (0, 2\pi) \times \mathbb{R}$
- **(Helicoid)** $\mathbf{x}_2(\theta, v) = (\sinh v \cos\theta, \sinh v \sin\theta, \theta), (\theta, v) \in (0, 2\pi) \times \mathbb{R}$

Note that Catenoid and Helicoid are both minimal surfaces, and so have mean curvature identically zero. However, mean curvature of two isometric surfaces may not be equal.

2.3 Existence of Surfaces for two given forms

This part is out of the scope of the course, so I will not deep talk for this subsection here. It is natural to ask whether giving arbitrary fundamental forms I and II , we can still find a surface whose fundamental forms are exactly as given? The answer is negative.

2.3.1 Example

Does there exist a parametrization $\mathbf{x}(u, v) : D \to \mathbb{R}^3$ of a surface S such that the first and second fundamental forms are given by:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad II = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

for any $(u, v) \in D$? Explain your answer.

Solution. From the formula of Theorem 3.5.4 in lecture notes, we may compute the Gaussian curvature of this surface by

$$K = -\frac{1}{2\sqrt{EG}} \left[\left(\frac{E_v}{\sqrt{EG}} \right)_v + \left(\frac{G_u}{\sqrt{EG}} \right)_u \right] = 0$$

but we see that

$$K = \frac{\det(II)}{\det(I)} = \frac{-1}{1} = -1 \neq 0$$

Thus, such surface S doesn't exist.

Note. However, if such set of fundamental forms satisfy the **Gauss-and-Codazzi equations**, another set of PDEs, then a surface can be defined with those forms. As this is out of our scope, let us stop here!

3 Total Curvature and Gauss-Bonnet Theorem

3.1 Compact Surface

A **compact** surface is a surface that can be obtained from a polygon, or a finite number of polygons by identifying edges.

In lecture notes, this surface is named **simple closed** surface. A **closed** surface is a surface that is *compact* and *without boundary.*

3.2 Euler Characteristics for Polyhedra Surfaces

The **Euler characteristic** of a compact polyhedra surface S is defined by

$$\chi(S) = V - E + F$$

where V, E, F are the numbers of vertices, edges and faces of a polyhedron modeled on S, respectively.

3.3 Classification of all compact connected surfaces

Let S be a compact connected surface in $\mathbb{R}^3$. Then,

- S is homeomorphic to a sphere with a number of handles g (genus) attached,
- $\chi(S) = 2(1 - g)$, where $g = 0, 1, 2, \ldots$

3.4 Gauss-Bonnet Theorem for Orientable surfaces

If $R \subset S$ be a regular region of an orientable surface and let $C_1, C_2, \ldots, C_n$ be the simple closed piecewise regular curves which form boundary of R, denoted by ∂R, and $\theta_1, \theta_2, \ldots, \theta_p$ be the exterior angles, then

$$\iint_R K\,dA + \sum_{i=1}^{n} \int_{C_i} \kappa_g(s)\,ds + \sum_{i=1}^{p} \theta_i = 2\pi\chi(R)$$

and the middle term of the left-hand-side conducts a contour integer, which is integrating along a closed curve, that is $C_1 \sqcup C_2 \sqcup \cdots \sqcup C_n = \partial R$.
Remark. If R is simple, then $\chi(R) = 1$.

3.5 Gauss-Bonnet Theorem for Orientable, Closed surface

For a **closed** surface has **no boundary**, we can omit the line integral and summation of the above subsection. If S is a closed orientable surface, then

$$\iint_S K\,dA = 2\pi\chi(S)$$

The left-hand-side of the above formula is called the **total curvature** of S. For geometrical meaning, you can think it as "adding all the K at each point of the surface"

I know the above contents are very abstract to you, and you may find it difficult to understand and visualize. Let me demonstrate some examples and you may get a feel on applying Gauss-Bonnet formula for a closed surface is enough.

3.5.1 Examples

1. Show that for any closed surface with everywhere positive curvature is *homeomorphic* to the sphere.

2. Let S be a closed surface not homeomorphic to the sphere. Show that K attains both positive and negative values.

4 Final Revision

The course is almost over! I know the course content comes a bit rush when the course is almost over. I wish you can still pick away something important related to "Differential Geometry" when you successfully persist in the end of the course!

Before we end up the last tutorial and going to the final examination, I want to summary all the main contents of our tutorial notes for you:

- **Ch1 - Preliminary on Calculus and Linear Algebra**
  - Definite and Indefinite Integration Methods
  - Using elementary row operations to find inverse of matrix
  - Common Proof techniques
  - Matrix Representation of Linear transformation
  - Calculus on vector-valued functions
- **Ch2 - Curves Theory, plane and space curves**
  1. Understanding "regular" curves, computing arc-length of curves
  2. Parametrization on regular plane and space curves
  3. Computing curvature and torsion for plane and space curves
  4. Applying Frenet Frame **T, N, B** and Frenet-Serret Equations .
- **Ch 3 - Surface Theory**
  1. Understanding "regular" surfaces, computing tangent space
  2. Computing 1st fundamental form, surface area, arc-length of curves on surface
  3. Computing 2nd fundamental form, gauss map & Gaussian curvature
  4. Understanding Shape operator, computing mean curvature, showing minimal surface
  5. Checking local Isometry, theorema egregium
  6. Simple applying of Gauss-Bonnet Theorem and compute $\iint_S K\, dA$

Question Bank

1. (a) Which of the following is/are regular curves? Provide a short explanation.
    - $\gamma : (-1, 1) \to \mathbb{R}^2,\ \gamma(t) = (t^2 - 1, t(1 + t^2))$,
    - $\gamma : (-1, 1) \to \mathbb{R}^3,\ \gamma(t) = (t^2, t^3, t^4)$ and
    - $\gamma : (0, 1) \to \mathbb{R}^2,\ \gamma(t) = (t^2 - 1, t^3)$.

   (b) Consider the parametrized curve $\gamma : \mathbb{R} \to \mathbb{R}^3$ given by

$$\gamma(t) = \left(\sqrt{1+t^2}, t, \ln\left(t + \sqrt{1+t^2}\right)\right)$$

   (i) Compute the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$.

   (ii) Find the curvature $\kappa(t)$ and torsion τ .

   (iii) Write $\mathbf{N}'(t)$ as a linear combination of $\mathbf{T}, \mathbf{N}, \mathbf{B}$ using the Frenet formula.

   (iv) Simplify the expression $(\mathbf{N}' \times \mathbf{B}') \cdot \mathbf{T}$.

2. Let $\alpha : I \to \mathbb{R}^3$ be a space curve parametrized by arc-length with $\kappa(s) > 0, \kappa'(s) \neq 0$ and $\tau(s) \neq 0$ for all $s \in I$. Prove that the trace of α is contained in a sphere of radius $r > 0$ if and only if

$$\frac{1}{\kappa(s)^2} + \frac{\kappa'(s)^2}{\kappa(s)^4 \tau(s)^2} \equiv r^2.$$

3. Let $X : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth map defined by

$$X(u, v) = (u, v, uv)$$

   Suppose that $S = X(U)$ is a surface.

   (a) Show that S is a regular parametrization.

   (b) Compute the first fundamental form I and the second fundamental form II of X with respect to the unit normal vector $\dfrac{X_u \times X_v}{\|X_u \times X_v\|}$.

   (c) Compute the Gaussian curvature K and the mean curvature H of S.

4. Let S be the surface parametrized by

$$\mathbf{x} : (-\pi, \pi) \times \mathbb{R} \to \mathbb{R}^3, \quad \mathbf{x}(u, v) = (\cosh v \cos u, \cosh v \sin u, v)$$

   With respect to this parametrization,

   (a) Define and compute the Gauss map of S.

   (b) Let $S_1 = \mathbf{x}(U)$, where $U = (-\pi, \pi) \times (0, 1)$. Compute the total surface area of S_1.

   (c) Show that S is a minimal surface.

   (d) Is there a surface $T \in \mathbb{R}^3$ which has the same first fundamental form as S? Explain your answer.

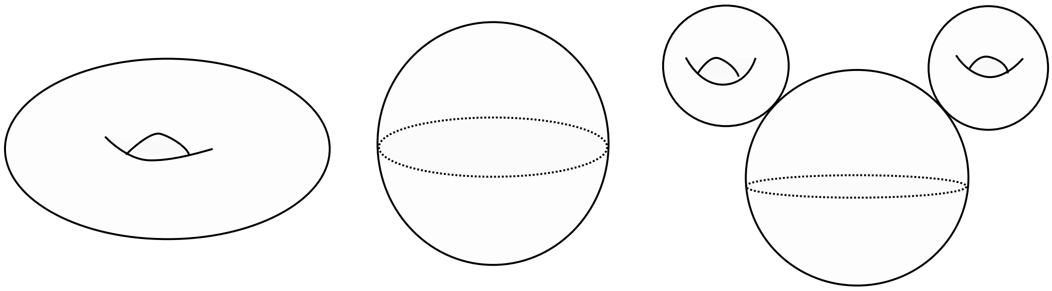

Figure 1: A torus, a sphere and a mickey mouse with two ears.

5. (2023,Final Q3)

   (a) State the genus and Euler characteristics of a torus.

   (b) State the genus and Euler characteristics of a sphere.

   (c) Dr. Cheng has a spherical bread. Each TA used some strawberry jam to stick a donut (torus) onto the bread. The final product looks like a Mickey mouse with many ears. State its genus and Euler characteristics in terms of the number of ears m. You can assume the surface is differentiable.

6. (2023, Final Q4) Consider the surface $M : \mathbf{x}(u, v) = (u, v, \cosh u), u, v \in \mathbb{R}$.

   (a) Show that the Gaussian curvature of M is 0.

   (b) Find the arc-length parametrization of $\gamma(u) = (u, \cosh u)$.

   (c) Using (b), show that M is isometric with the plane $\mathbb{R}^2$.

   (d) Clive claims that we have another proof: Using (a), Gaussian curvature of M and the plane $\mathbb{R}^2$ are both 0, so by Theorema Egregium they are isometric. Do you think Clive is correct? Explain.

7. (a) Is it possible that the mean curvature H of a compact surface S is entirely zero? Explain your answer.
   (**Note.** This want to show there exist no compact minimal surfaces in $\mathbb{R}^3$.)

   (b) (2018, Final Q4) Let M be a smooth closed surface with Gaussian curvature K.

      (i) If $K < 0$ everywhere on M, what is the minimum genus of M?

      (ii) If $K < -4$ entirely on M and the Euler characteristic of M is $\chi(M) = -6$, show that the surface area of M is not more than 3π.